Math 142 Lecture 7 Notes

Daniel Raban

February 6, 2018

1 Compactness and Connectivity in $\mathbb{R}^n$

1.1 The Heine-Borel theorem and compactness in $\mathbb{R}^n$

Theorem 1.1 (Heine-Borel)**.** *Any closed and bounded interval $[a, b] \subseteq \mathbb{R}$ is compact.*

Proof. Give $[a, b] \subseteq \mathbb{R}$ the subspace topology, and let $\mathcal{C}$ be an open cover of $[a, b]$. Let $X = \{x \in [a, b] : [a, x]$ is contained in the union of finitely many elements of $\mathcal{C}\}$. If $b \in X$, then $[a, b] = U_1 \cup \cdots \cup U_n$ for $U_i \in \mathcal{C}$, so $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{C}$.

Think of $X \subseteq \mathbb{R}$. We know that $a \in X$, and $[a, a] = \{a\}$ is contained in some $U \in \mathcal{C}$ such that $a \in U$. Additionally, X is bounded above by b. So X has a supremum $s \in \mathbb{R}$. We want to show that $s = b$ and that $s \in X$.

Certainly, $s \leq b$, so $s \in [a, b]$. Let $U \in \mathcal{C}$ be an open set such that $s \in U$. If $s < b$, then we can find some $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq U$. If $s = b$, then we can find some $\varepsilon > 0$ such that $(s - \varepsilon, s] \subseteq U$; this set is also open in the subspace topology on $[a, b]$. We can find points of X arbitrarily close to s; i.e. we can find $x_\varepsilon \in X$ such that $|s - x| < \varepsilon/2$. If $x_\varepsilon \in X$, then $[a, x_\varepsilon] \subseteq U_1 \cup \cdots U_n$ for some $U_i \in \mathcal{C}$. if $s < x_\varepsilon$, then $[a, s] \subseteq [a, x_\varepsilon]$, so $s \in X$. If $s > x_\varepsilon$, then $[x_\varepsilon, s] \subseteq U$. So $[a, s] \subseteq U_1 \cup \cdots \cup U_n \cup U$, which makes $s \in X$.

Also, if $s < b$, then $[a, s + \varepsilon/2] \subseteq U_1 \cup \cdots \cup U_n \cup U$. So $s + \varepsilon/2 \in X$, contradicting the fact that s is the supremum of X. So $s = b$, which shows that $\mathcal{C}$ has a finite subcover. Since $\mathcal{C}$ was arbitrary, we conclude that $[a, b]$ is compact. $\square$

This implies the following theorem, which is more our end-goal.

Theorem 1.2. *$A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.*

Proof. ($\implies$) We proved this last lecture.

($\impliedby$) A is bounded, so $A \subseteq [-s, s]^n$ for some $s > 0$. Let $C = [-s, s]^n$. The set $[-s, s]$ is compact in $\mathbb{R}$ by our previous theorem, so our product theorem for compact spaces says that $C \subseteq \mathbb{R}^n$ is compact. Then $A \subseteq C$ is closed in the subspace topology. As a closed subset of a compact space, A is compact. $\square$

1.2 Connectivity

Definition 1.1. A space X is *connected* if whenever $X = A \cup B$ with A, B open and $A \cap B = \varnothing$, then either $A = \varnothing$ or $B = \varnothing$.

Here are a few equivalent definitions:

1. If $X = A \cup B$ with A, B open and nonempty, then $\overline{A} \cap B = \varnothing$ or $A \cap \overline{B} = \varnothing$.

2. If $A \subseteq X$ is both open and closed, then $A = X$ or $A = \varnothing$.

3. If $A \subseteq X$ has empty boundary, then $A = X$ or $A = \varnothing$.

4. If $f : X \to \{1, 2\}$ is continuous, and $\{1, 2\}$ has the discrete topology, then f is constant.

Theorem 1.3. $\mathbb{R}$ *is connected.*

Proof. If $\mathbb{R} = A \cup B$ with A, B open and $A \cap B = \varnothing$, then $\mathbb{R} \setminus A = B$ and $\mathbb{R} \setminus B = A$ are closed. Choose $x \in A$ and $y \in B$, and assume (without loss of generality) that $x < y$. Let $X = \{b \in [x, y] : [b, y] \subseteq B\}$. We know $y \in B$ and $y \in [x, y]$, so $y \in X$, making $X \neq \varnothing$. Also, x is a lower bound for X. So $I = \inf X \in \mathbb{R}$ exists. As the infimum of X, I is a limit point of X. Since $X \subseteq B$, I is a limit point of B, so $I \in \overline{B} = B$. This means $I \notin A$. Since B is open, we can find $\varepsilon > 0$ such that $(I - \varepsilon, I + \varepsilon) \subseteq B$. So $[I - \varepsilon/2, y] \subseteq B$, contradicting the definition of I as the infimum of X. □

Theorem 1.4. *A nonempty $X \subseteq \mathbb{R}$ is connected iff X is an interval (i.e. $X = (a, b)$ or $[a, b]$ or $(a, b]$ or $[a, b)$).*

Proof. ($\Longleftarrow$) This is the same proof as the previous theorem.

($\Longrightarrow$) If X is connected but X is not an interval, then there exist $a, b \in X$ and $p \in \mathbb{R} \setminus X$ such that $a < p < b$. Let $A = \{x \in X : x < p\}$, and let $B = \{x \in X : x > p\}$. Then $A, B \neq \varnothing$, as $a \in A$ and $b \in B$. We have $X = A \cup B$ and $A \cap B = \varnothing$, as $x \in X$ satisfies either $x < p$ or $x > p$. To show that A is open, we show that B is closed. Since $p \notin X$, $\overline{B} \subseteq X$ only contains points larger than p; so $\overline{B} = B$. This means that B is closed, so A is open. Similarly, A is closed, so B is open. This contradicts X being connected. □